temperature employed in the calculations. It is presumed that such information is covered adequately in Russian references cited.

—CHARLES W. BAULKNIGHT
Space Sciences Laboratory
General Electric Company

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Hypersonic Area Rule

M. D. LADYZHENSKII

In an earlier work[1] on the hypothesis[2] that the whole mass of gas is concentrated in an infinitely thin layer contiguous to the shock wave, a hypersonic area rule was formulated. According to this rule, when there is a flow past thin blunted nonaxisymmetrical bodies which have equal quantities of bluntness resistance and the same rules of variation in the direction of flow of the cross-sectional areas, and of the surface of the shock waves, the rules of pressure change and consequently the forces of resistance acting on the body as well, coincide, in which case the surfaces of the shock waves have an axial symmetry.

In the present work the limits of applicability of the results of Ref. 1 are established, and the hypersonic area rule is made more accurate by means of the introduction of an entropy layer.

1. Determination of the Limits of Applicability of the Results of Ref. 1

As an example of the application of the hypersonic area rule,[1] we construct a body equivalent to a thin round cone, i.e., one having the same quantity of bluntness resistance as a cone and the same trend of change in cross-sectional area in the direction of flow. The cross section of the body is postulated as having the shape of an ellipse, the major semiaxis of which is equal to the radius of the shock wave, and the area equal to the area of the cross section of the round cone (Fig. 1). In other words, the eccentricity of the ellipse in each section has its maximum possible value compatible with the requirement (condition 3 of Ref. 1) according to which the body should not go beyond the limits of a volume confined to the surface of the shock wave.

As was mentioned in the earlier work,[1] the area rule may be combined with the similarity law when the flow occurs past thin blunted bodies,[2] as a result of which the dimensionless quantities characterizing the flow are determined, for a fixed value of the adiabatic index $\varkappa$, by two dimensionless parameters: the known parameter of similarity when the flow is past thin blunt bodies $K = M_\infty\tau$ and the parameter $K_1 = (\pi/2c_xS)^{1/2}L\tau^2$, characterizing the influence of the bluntness, which in its order of magnitude is equal to the square root of the ratio of the resistance of the body to the resistance of the bluntness. Here $\tau = S^{1/2}/L$ is the small dimensionless quantity characterizing the thickness of the body; S some characteristic cross-sectional area of the body; L the length of the body; c_x and S, respectively, are the coefficient of bluntness resistance and the midships area of the bluntness.

Under the hypothesis that the action of bluntness may be replaced by the effect of an explosion in the leading point of the body with an energy equal to the bluntness resistance, the shape of the bluntness is nonessential.[1] In view of this, the bluntness area is introduced in the expression for k_1 instead[2] of its diameter. Let us assume that the number M_∞ of unperturbed flow equals infinity. Then for a fixed $\varkappa$ the dimensionless variables will depend on the single parameter K_1.

In Fig. 2 are shown (for the case of $\varkappa = 1.4$) the relations of the quantities k (ratio of major to minor semiaxis of the ellipse) and $(X - X_0)/X_0$ (the ratio of the resistance of the body, after the deduction of the bluntness resistance, to the bluntness resistance) in function $K_1 = (\pi/2c_xS)^{1/2}L \tan^2 \alpha$, where α is the angle of the half-aperture of the round cone. The shape of the shock wave was determined from the solution of the problem of a flow past a thin blunt cone according to Ref. 2. The area rule has a significance in use where $X/X_0 \geqq 1.1$, corresponding (see Fig. 2) to $K_1 \geq 0.1$. For low values of K_1 the resistance of the body is practically determined by the magnitude of the bluntness resistance. For large values of K_1 the area rule loses its force when k approaches unity, or more accurately[1] when $k - 1 \sim (\varkappa - 1)/(\varkappa + 1)$, which takes place approximately when $K_1 = 1.2$.

Thus the range of applicability of the area rule falls within the limits of $0.1 \leqq K_1 \leqq 1.2$. Here the ellipse in the cross section of the body may have a fairly elongated shape, different from a circle ($13 \geqq k \geqq 1.3$). This result may be of practical interest. However, in view of the fact that the results of Ref. 1 are obtained under rough assumptions of the concentration of the whole mass of gas in an infinitely thin

Translated from Inzhenernyi Zhurnal (Engineering Journal) 1, No. 1, 159–163 (1961). Translated by Singer, Smith, and Co., New York.

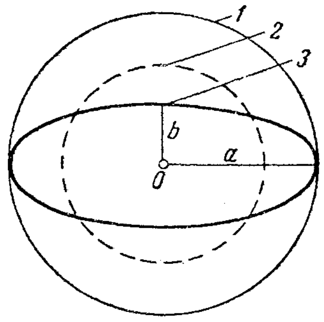

Fig. 1 1) Shock wave; 2) cross section of spherical body; 3) cross section of equivalent body.

layer beyond the shock wave, it is necessary to make the area rule more accurate.

2. Improving the Accuracy of the Area Rule

In the first place, we assume $M_\infty \geqq 1$, but in contrast to Ref. 1, we do not impose the necessary condition of $M_\infty \tau \geqq 1$. We introduce a cylindrical system of coordinates xL, yL, θ (the x axis passes through the leading point of the body and is directed along the flow). We designate as uU_∞, vU_∞, and wU_∞ the components of the velocity in an axial, radial, and circumferential direction to the axis, respectively: $p\rho_\infty U^2_\infty$ is the pressure, $\rho\rho_\infty$ is the density, and U_∞ is the velocity of the unperturbed flow directed along the x axis, and ρ_∞ is its density. We shall characterize the quantity of bluntness by the dimensionless diameter of bluntness dL, where d is a small quantity. We write the equation for the surface of the body in the form $y = \tau f(x,\theta)$.

We distinguish the entropy layer, i.e., the region occupied by flow lines that have passed the portion of the surface of the shock wave which has passed, where the angles of inclination formed by the surface of the shock wave with the direction of the unperturbed flow are not small (Fig. 3). We let the equation of the conditionally introduced boundary of the entropy layer be $y = \delta\Phi(x,\theta)$, where δ is a small quantity. Starting from some $x = x_0 \sim d$, the angles of inclination of the boundary of the entropy layer with respect to the x axis will be of the order of δ.

We next present evaluations of the parameters of flow in an entropy layer analogous to those presented in another work.[3] We note also that the influence of the entropy layer on the distribution of pressure along the thin blunted cone is considered in Ref. 4.

We assume that at the surface of the entropy layer the relation $p \sim d^\alpha$ exists, where α is a positive number to be determined.

As is found further on, the order of the pressure across the entropy layer is maintained, since for densities, by using adiabatic conditions, we may write $\rho \sim d^{\alpha/\varkappa}$. We now write the equation of continuity for the entropy layer. Comparing the discharge in the entropy layer with the discharge in the jet

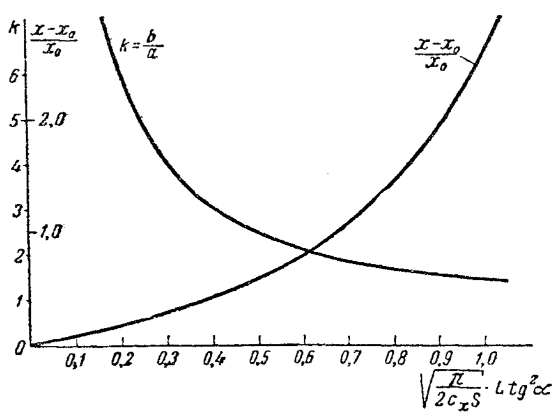

Fig. 2.

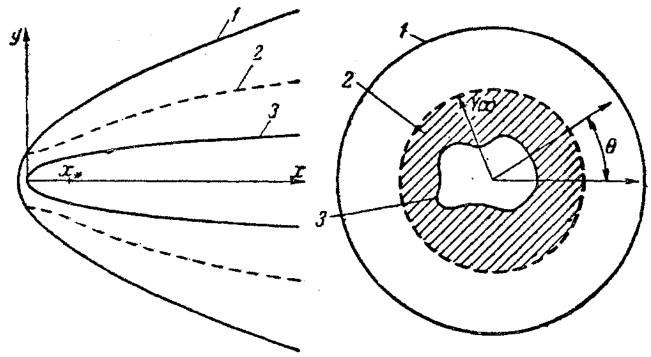

Fig. 3 1) Shock wave; 2) entropy layer; 3) body.

stream of the unperturbed flow through an area equal to the area of the bluntness midships, we have

$$d^2 \sim \rho u \sigma \tag{1}$$

where σ is the area occupied by the entropy layer in section x = constant (hatched in Fig. 3). Since it follows from the equation of Bernoulli that in the entropy layer $u \sim 1$, we have $\sigma \sim d^{[2-(\alpha/\varkappa)]}$. It is evident that for δ, which enters into the equation of the boundary of the entropy layer, we obtain the definition

$$\delta^2 = S + \sigma \tag{2}$$

where S is the cross-sectional area of the body. We require that the area of the body, in its order of magnitude, should not exceed the area of the entropy layer $S \lesssim \sigma$; then, evidently

$$\delta^2 \sim d^{[2-(\alpha/\varkappa)]} \tag{3}$$

Since the usual expression for hypersonic flow $p \sim \delta^2$ is correct for the pressure, this yields the equation for the determination of α:

$$d^\alpha \sim d^{2-(\alpha/\varkappa)} \qquad \alpha = \frac{2\varkappa}{\varkappa + 1} \tag{4}$$

Finally, for the parameters of flow in the entropy layer we have:

$$\begin{aligned} p &\sim d^{[2\varkappa/(\varkappa+1)]} & \rho &\sim d^{[2/(\varkappa+1)]} \\ \delta &\sim d^{[\varkappa/(\varkappa+1)]} & \sigma &\sim d^{[2\varkappa/(\varkappa+1)]} \end{aligned} \tag{5}$$

on the condition that τ and d are related in order of magnitude by the expression derived from the condition $S \sim \sigma$:

$$\tau \sim d^{[\varkappa/(\varkappa+1)]} \tag{6}$$

this relation practically coincides with the condition $\tau \sim \sqrt{d}$, which expresses the fact that the resistance of the body is comparable in order of magnitude to the bluntness resistance.[2]

We shall evaluate the pressure drop in the entropy layer. From equations of motion, we have

$$\frac{\partial p}{\partial y} = -\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial y}{\partial y} + \frac{w}{y}\frac{\partial v}{\partial \theta}\right) \tag{7}$$

$$\frac{1}{y}\frac{\partial p}{\partial \theta} = -\rho\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \frac{w}{y}\frac{\partial w}{\partial \theta}\right)$$

Consequently $(\partial p/\partial y) \sim (1/y)(p/\theta)$, since all the terms on the right sides of Eq. (7) are of the same order of smallness. From this, taking into account (5) and (6) for the pressure drop in both a radial and a circumferential direction, the following estimate is valid:

$$\Delta p \sim d^2 \sim \tau^{[2(\varkappa+1)]/\varkappa}$$

Thus the pressure in the entropy layer may be considered constant with a relative error of

$$\Delta p/P \sim d^{[2/(\varkappa+1)]} \sim \tau^{2/\varkappa} \tag{8}$$

which is somewhat greater than the relative error in the theory of small perturbations of hypersonic flow, which is known to equal τ^2.

Let us formulate the problem of a flow past (a body—Transl.). With $x < x_0$, assume an axisymmetrical nose part of the body with an axis of symmetry directed along the x axis, the flow around which has been calculated fully. It follows from what has been said, inasmuch as the entropy layer cannot maintain the pressure drop in a circumferential direction, in the region $x > x_0$ at the outer boundary of the layer, the pressure in the section x = constant should be constant. For this condition to hold it is sufficient that the surface bounded by the entropy layer should possess an axial symmetry. [Its equation in this case may be written as $y = \delta Y(k)$.]

Then the flow outside of the entropy layer, which is axially symmetrical by the condition where $x < x_0$, retains its axial symmetry for $x > x$ as well, and consequently the condition of constancy of pressure in the circumferential direction at the outer boundary of the entropy layer will be satisfied.

We now introduce an equation which relates S to σ where $x > x_0$, for which, analogously to Ref. 3, we make use of the equation of continuity. Let us designate with the subscript 0 the quantities in the plane x_0. Differentiating the elemental jet of the flow in the entropy layer, we write for it an equation of discharge:

$$\rho_0 u_0 y_0 d\theta_0 dy_0 = \rho u y d\theta dy \tag{9}$$

From the adiabatic and Bernoulli equations, in which terms of the order of τ^2 are discarded, we have the following for ρ and u:

$$\rho = \rho_0 \left(\frac{p}{p_0}\right)^{1/\varkappa}$$

$$\frac{u^2}{2} + \frac{\varkappa}{\varkappa - 1}\frac{p_0}{\rho_0}\left(\frac{p}{p_0}\right)^{[(\varkappa - 1)/\varkappa]} = \frac{1}{2} + \frac{1}{(\varkappa - 1)M_\infty^2} \tag{10}$$

We analyze Eq. (9) for ρu and integrate for the whole area of the entropy layer in the section $x = x_0$ [we take advantage of the fact that the coordinates of the line of flow chosen, y and θ, satisfy the relations $y = y(y_0,\theta_0)$, $\theta = \theta(y_0,\theta_0)$]. On the right side of the equation we evidently obtain the area σ occupied by the entropy layer in the section x. When we take into account the axial symmetry of the boundary of the entropy layer, we have $\sigma = \pi\delta^2 Y^2(x) - S$. Finally the relation sought is written:

$$\pi\delta^2 Y^2(x) - F(p) = S(x)$$

$$F(p) = \iint_{\sigma_0} \left(\frac{p_0}{\rho}\right)^{1/\varkappa} \sqrt{\frac{1 + [2/(\varkappa - 1)](1/M^2_\infty) - [2\varkappa/(\varkappa - 1)](p_0/\rho_0)}{1 + [2/(\varkappa - 1)][1/M^2_\infty] - [2\varkappa/(\varkappa - 1)](p_0/\rho_0)(p/p_0)^{[(\varkappa - 1)/\varkappa]}}}\, y_0 d\theta_0 dy_0 \tag{11}$$

The axisymmetrical flow beyond the entropy layer, where $x > x_0$, may be calculated by one of the exact methods, for example, by the method of characteristic curves. The boundary of the entropy layer will be determined here in the process of solution from Eq. (11), which plays the part of a boundary condition, replacing the condition of nonflow. From this it follows that the flow where $x > x_0$ is determined fully by the given law of change in the cross-sectional area of the body $S(x)$. In this case, one more obvious limitation should be imposed on the shape of the body: the body should not go beyond the limits of the "entropy circle" (see Fig. 3), which may be symbolically written as

$$S \subset \pi\delta^2 Y^2(x) \tag{12}$$

Now it is possible to formulate a more accurate hypersonic area rule. When a flow occurs past thin blunted bodies having axisymmetrical nose parts coinciding at some distance from the leading point of the body, and the same trend (or pattern—Transl.) of change in the cross-sectional areas of the remaining parts, then:

a) The flows outside of the entropy layers are axisymmetrical; the parameters of flow at the corresponding points of the surface of the shock waves and of the conditionally introduced boundaries of the entropy layers coincide.

b) The pressure in the entropy layers depends solely on x, and the pattern of the change in pressure is the same for the bodies in question; because of this the forces of resistance acting on the body are equal, since the resistance X is expressed in the form

$$X = X_0 + L^2\rho_\infty U_\infty^2 \int_{x0}^{1} S'(x)p(x)dx \tag{13}$$

where X_0 is the resistance of the nose portion of the body. For this it is assumed that the conditions of Eqs. (6) and (12) are satisfied.

The results obtained are generalized without difficulty to the case of flow with dissociation. Making allowance for these effects leads only to a change in the form of the function $F(p)$ in Eq. (11).

3. Comparison of Results

Let us compare the result obtained with the area rule demonstrated in Ref. 1. The two theorems have in common the requirement of a coincidence in the laws (or patterns—Transl.) of change in the cross-sectional area in the direction of the x axis, as well as the condition according to which the resistance of the body should not exceed in its order of magnitude the bluntness resistance. The difference from the formulation of the theorem demonstrated in Sec. 2 consists of the following:

a) The necessary condition $M_\infty\tau \geqq 1$ is not imposed as it was in Ref. (1).

b) Instead of the requirement of equal quantities of bluntness resistance,[1] a stronger limitation is imposed: The nose parts of equivalent bodies, being axisymmetrical, should coincide in shape.

c) Instead of condition (3) of Ref. 1, according to which the body does not go beyond the volume limits set by the surface of the shock wave, the stronger limitation of Eq. (12) is imposed, according to which the body should not exceed the limits of the outer boundary of the entropy layer.

As a result of this, it may be expected that the values of k found in Sec. 1, which characterize the difference between the cross section of the body and the cross section of an equivalent body of revolution, are somewhat excessive.

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